A Dynamic Investigation of Sucker-Rod Pumping

by Roy M. Knapp

B.S., University of Kansas, 1963

Professor in Charge
Floyd W. Preston

Submitted to the Department of Chemical and Petroleum Engineering of the University of Kansas in partial fulfillment of the requirements for the Degree of Master of Science.

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JUNE ~~January~~, 1969

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CHAPTER I

INTRODUCTION

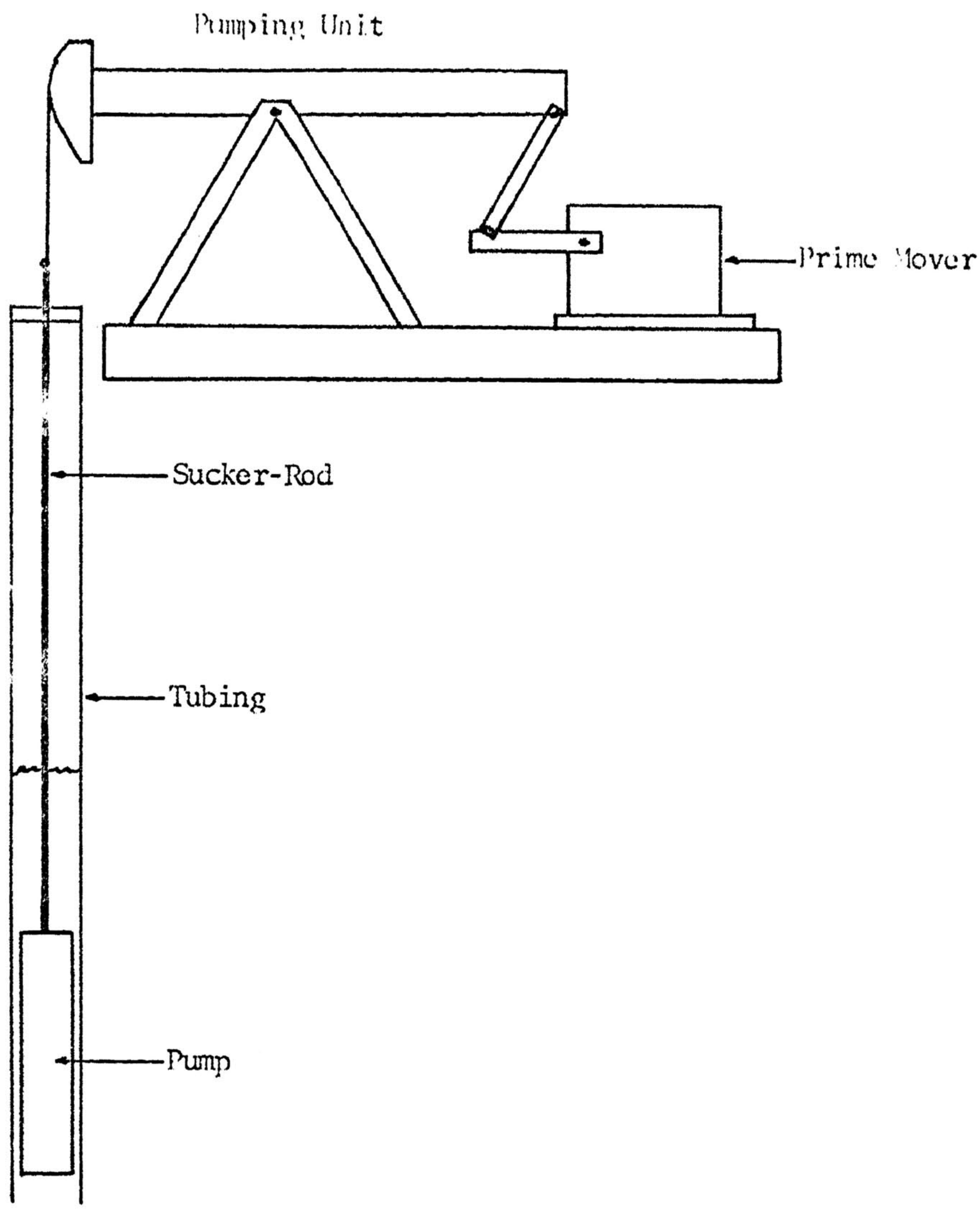
Background

In the past, field studies of sucker-rod pumping systems have yielded only limited performance information. It is possible to readily measure displacements and force at the surface. However, directly obtaining displacement and force data below the surface is a difficult and expensive task. Since sucker-rods pump approximately ninety percent of all wells using artificial lift, it is important to understand the behavior of such a system. Sucker Rod Pumping Research, Inc.,[1] has utilized a mechanical simulator and analog computer to simulate sucker-rod pumping systems. Gibbs[2] has used digital computers to simulate dynagraph cards for pumping systems. This paper develops a mathematical model of a simplified pumping system to demonstrate its validity in simulating a pumping system. It then utilizes a modification of the model and surface data to predict pump behavior. The predicted pump behavior can be used to evaluate the efficiency of the pumping system.

The mechanical system of sucker-rod pumping does not lend itself readily to exact mathematical formulas. Before the presence of high speed digital computers, any attempt to investigate dynamic behavior would have been impossible. The time involved in finite difference methods or the large amount of expensive experimental data would have made such studies prohibitive.

The Sucker-Rod Pumping System

Figure 1. Schematic-diagram of Sucker-Rod Pumping System



The pumping system shown in Figure 1 consists of six basic units:

1. A prime mover
2. A pumping unit to convert rotary motion to rectilinear motion
3. A long sucker-rod to transmit the rectilinear motion from the pumping unit to the pump
4. A pump at the bottom of the string
5. A tubing string
6. The column of fluid being pumped

Each of these elements is quite simple. The dynamic characteristics of each can be readily determined. However, when all six elements are interacting, the behavior becomes quite complex. To simplify the system simulated, the effects of several elements were neglected. The prime mover was considered to run at constant speed. The inertial effects of the pumping unit and the counter weights were neglected. The sucker-rod string was assumed to be a single diameter. The pump operated at one hundred percent efficiency. The tubing string was assumed to be anchored so there was no movement of the pump barrel and standing valve relative to static conditions.

As stated above, the primary purpose of this paper is to demonstrate the feasibility of using finite mathematical models to simulate and analyze a pumping system. The neglected effects should be incorporated into future studies of pumping systems.

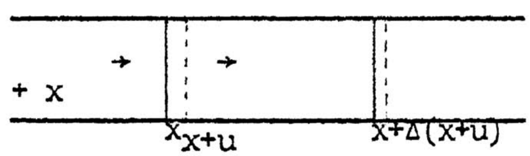
CHAPTER II

FORMULATION OF ANALYTIC EQUATIONS

Equation of Motion

The motion of a sucker-rod string can be represented by the one-dimensional wave equation. A viscous damping term is added to allow for pump damping effects. In this simulation, nonviscous effects such as coulomb friction and hysteresis losses are considered negligible. The wave equation is a boundary value problem requiring a set of boundary conditions. These involve the pump behavior and the surface unit motion. Initial conditions to determine initial stress and velocity complete the necessary conditions for a solution.

The wave equation describes the longitudinal vibrations of bar. Consider "x" as the distance along the axis of a long slender rod.



The term (x+u) then denotes the position at time "t" of the cross-section whose position is "x" when undisturbed. Therefore, "u" represents displacement. In differential terms, the element of length Δx is altered to $\Delta(x+u)$ or $(1+\frac{\partial u}{\partial x})\Delta x$. Using notation common to the field of strength of materials, the change in length of an element of unit length is $(1+\varepsilon)\Delta x$, where ε is the strain. Therefore, $(1+\frac{\partial u}{\partial x})\Delta x = (1+\varepsilon)\Delta x$ or,

$$\varepsilon = \frac{\partial u}{\partial x} \quad . \tag{1}$$

The tension forces across the cross-section A_{rod} is $\sigma \cdot A_{rod}$ where σ is the stress in the rod. Young's modulus of elasticity is defined as the ratio of stress to strain or $E = \frac{\sigma}{\varepsilon}$. Therefore,

$F_t = \sigma A_{rod} = E \cdot A_{rod}\, \varepsilon = E \cdot A_{rod} \frac{\partial u}{\partial x}$. The rate of momentum change due to the acceleration of the mass between x and Δx is $\rho A_{rod}\, \Delta x \frac{\partial^2 u}{\partial t^2}$. The sum of the forces between x and $x + \Delta x$ is equal to the rate of momentum change of the mass:

$$E \cdot A_{rod} \frac{\partial u}{\partial x}\bigg|_{x+\Delta x} - E \cdot A_{rod} \frac{\partial u}{\partial x}\bigg|_{x} = \rho\, A_{rod}\, \Delta x \frac{\partial^2 u}{\partial t^2} \qquad (2)$$

Dividing by Δx and taking the limit as $\Delta x \to o$ gives:

$$\rho\, A_{rod} \frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x \to o} \frac{E \cdot A_{rod} \frac{\partial u}{\partial x}\Big|_{x+\Delta x} - E \cdot A_{rod} \frac{\partial u}{\partial x}\Big|_{x}}{\Delta x} \qquad (3)$$

$$\rho\, A_{rod} \frac{\partial^2 u}{\partial t^2} = E \frac{\partial}{\partial x} \left(A_{rod} \frac{\partial u}{\partial x} \right). \qquad (4)$$

If the rod has a constant cross-section, this can be reduced to:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}. \qquad (5)$$

This is the familiar one-dimensional wave equation.

The wave equation describes forced vibration in a rod. But the sucker-rod pumping system has a pump that damps the vibrations. A viscous damping term involving $\frac{\partial u}{\partial t}$ is added to simulate this pump damping. For dimensional homogeneity the units of the viscous term must be length/time^2. Multiplication by a characteristic velocity and division by a characteristic length will make the equation dimensionally homogeneous. A characteristic velocity is the velocity of force propogation or $\sqrt{E/\rho}$ and a characteristic length is the fundamental wave length of the rod system ($4L/2\pi$). The damping term then becomes $\left(\frac{\sqrt{E/\rho}}{2L} \pi\, \nu \frac{\partial u}{\partial t} \right)$ where ν is a dimensionless coefficient of viscous damping. The equation of motion now becomes:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\pi\sqrt{E/\rho}}{2L}\,\nu\,\frac{\partial u}{\partial t} = \frac{E}{\rho}\,\frac{\partial^2 u}{\partial x^2}\,. \tag{6}$$

The equation assumes that all energy losses are contained in the viscous friction term. Coulomb friction and hysteresis are negligible. Equation (6) assumes a constant rod cross-section. However, tapered strings would require only slight modification. Gravity terms have not been mentioned above. Because a gravity term would appear in the differential equation as a static term, its effect would be constant. Therefore, the gravity effect was superimposed on the dynamic solution.

Boundary Conditions

For solution, Equation (6) needs four boundary conditions. Since the equation is second order in both time and axial distance, two boundary conditions are required for each independent variable.

Time Conditions

Time conditions are satisfied by specifying the initial stress and velocity of the rods. Initially the rods are at rest and the initial velocity is zero or $\left.\frac{\partial u}{\partial t}\right|_{x,o} = o.$ (7)

The initial stress in the rods is only the static stress and dynamic stresses are zero. In algebraic notation: $\sigma = o$ but

$$\sigma = \varepsilon E \text{ and } \varepsilon = \frac{\partial u}{\partial x}$$

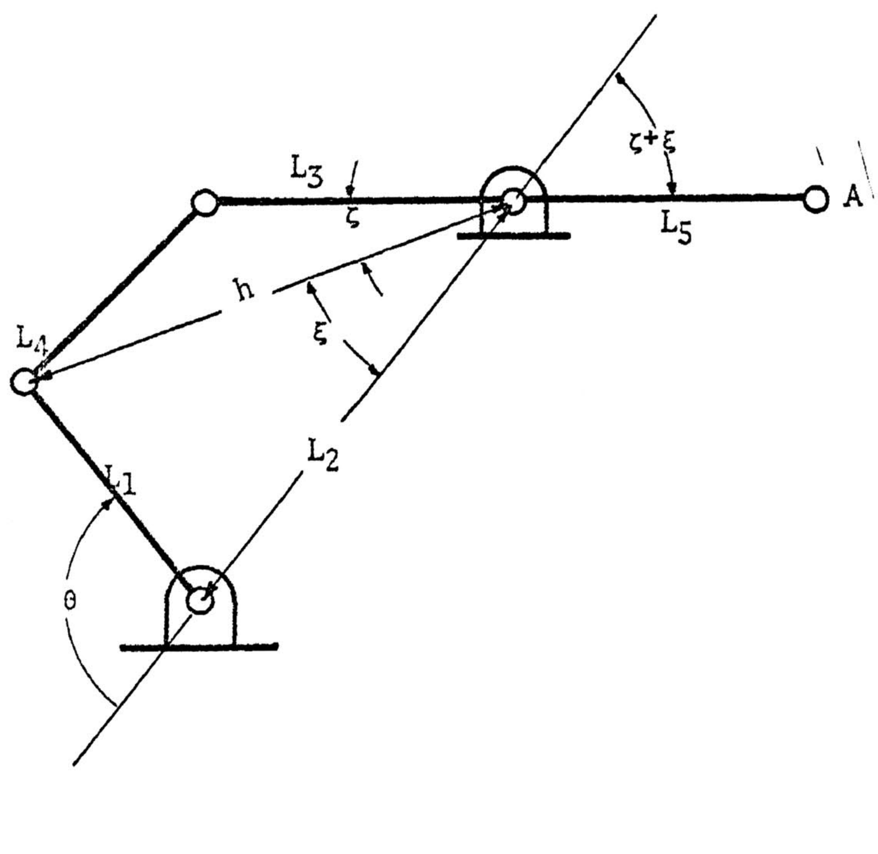
$$E\frac{\partial u}{\partial x} = o \quad \text{since } E \text{ is constant}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x,o} = o \tag{8}$$

Surface Condition

The surface boundary condition is specified by the motion of the polished rod. This motion is controlled by the geometry of the pumping unit and the torque characteristics of the prime mover. In the computer program the prime mover is assumed to move at constant speed. This assumption is satisfactory for single cylinder internal combustion engines with their large flywheels or for synchronous electric motors. However, for multi-cylinder engines or other electric motors, the torque variant speed should be considered.

Figure 2. Geometry of Surface Pumping Unit



The surface pumping unit shown in Figure 2 is a four-bar mechanism and its movement is solved by trigonometric considerations. The movement of $u_{o,t}$ or "A" can be written as a function of θ

$$A(\theta) = L_5 \ (\xi+\zeta) \tag{9}$$

The angles ξ and ζ are functions of θ.

$$\xi = \sin^{-1} \frac{(L_1 \sin \theta)}{h} \tag{10}$$

From the Law of Cosines:

$$L_4^2 = L_3^2 + h^2 - 2L_3h \cos \zeta$$

$$\cos \zeta = \frac{L_3^2 + h^2 - L_4^2}{2L_3h}$$

$$\zeta = \cos^{-1} \frac{(L^2{}_3 + h^2 - L_4^2)}{2L_3h} \tag{11}$$

Then: $$A(\theta) = L_5 \left[\sin^{-1} \frac{(L_1 \sin \theta)}{h} + \cos^{-1} \frac{(L^2{}_3 + h^2 - L_4^2)}{2L_3h} \right]. \tag{12}$$

Where: h is the distance between the saddle bearing center to the pitman bearing center.

$$h^2 = [L_1 \sin(\pi-\theta)]^2 + [L_2 - L_1 \cos(\pi-\theta)]^2$$

$$h^2 = [L_1^2 \sin^2(\pi-\theta) + L_2^2 - 2L_1L_2 \cos(\pi-\theta) + L_1^2 \cos^2(\pi-\theta)]$$

but $1 = \sin^2 \emptyset + \cos^2 \emptyset$ and

$$\cos \emptyset = -\cos(\pi - \emptyset).$$

Therefore, $h^2 = L_1^2 + L_2^2 + 2L_1L_2 \cos\theta$

$$h = \sqrt{L_1^2 + L_2^2 + 2L_1L_2 \cos\theta} \qquad (13)$$

For a constant speed prime mover, θ is easily expressed as a function of time.

$$\theta = 2\pi(SPM)\frac{t}{60} \text{ radians} \qquad (14)$$

Pump Condition

Before writing the pump boundary condition, the motion of the pump should be clearly understood. In Figure 3 the pump is at its lowest position. At this point the standing valve is closed and the traveling valve open.

The pump begins to ascend and the traveling valve closes just past point D. The standing valve remains closed until the gas in the pump expands enough to lower the internal pressure of the pump below the well pressure. During this part of the stroke, the fluid load is being picked up by the plunger and the rods as shown in Figure 4. At point A the well pressure exceeds the pressure in the pumps, the standing valve is forced open, and the liquid is admitted to the pump. While the plunger travels from A to B the rods bear the full weight of the fluid column above the pump, Figure 5.

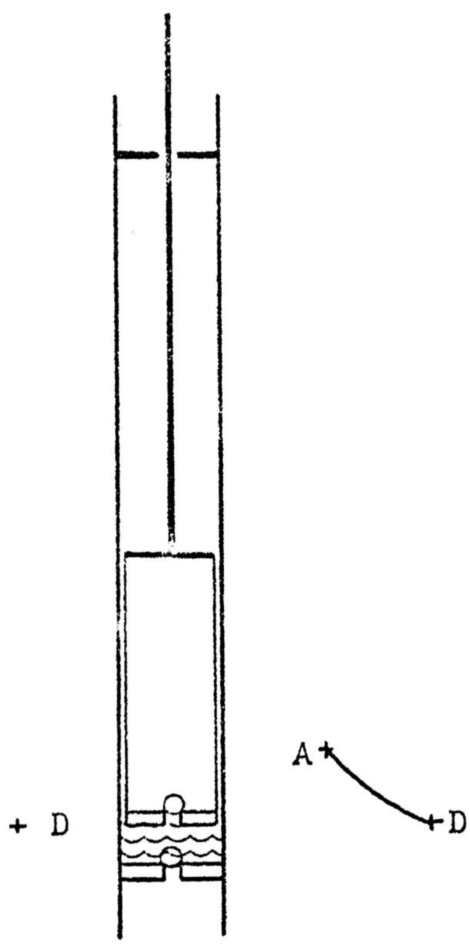


Figure 3.

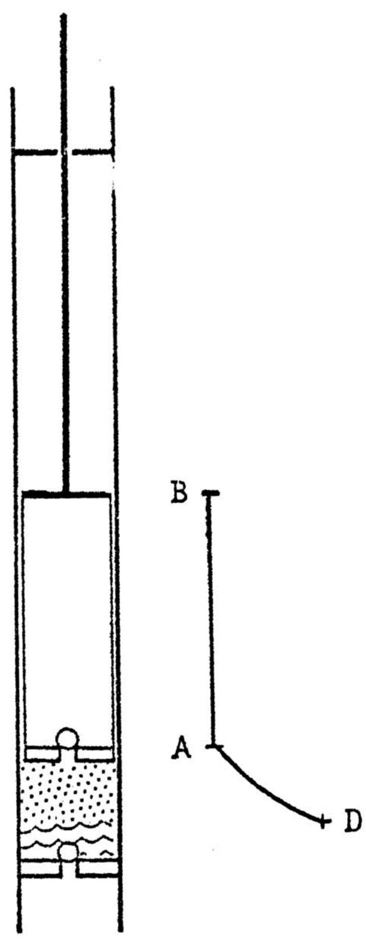


Figure 4.

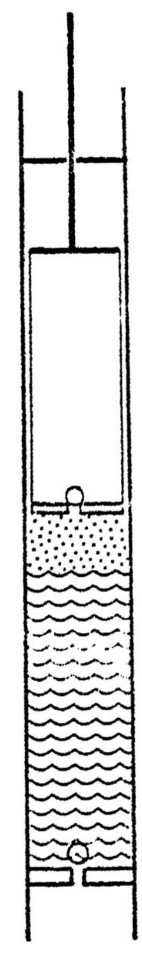


Figure 5.

Point B is the highest position the plunger reaches. As the plunger passes B the standing valve closes. The traveling valve remains closed as the pressure above the plunger exceeds the pressure below the plunger. As the pump descends, Figure 6, the gas in the pump is compressed and the fluid load is transferred to the tubing. At point C the pressure in the pump exceeds the fluid pressure above the plunger and the traveling valve opens. At this point the weight of the fluid column is borne by the tubing string. As the pump travels downward from C to D, Figure 7, the pump fluid passes freely through the traveling valve and into the tubing above the plunger. When the pump reaches D, the cycle is completed.

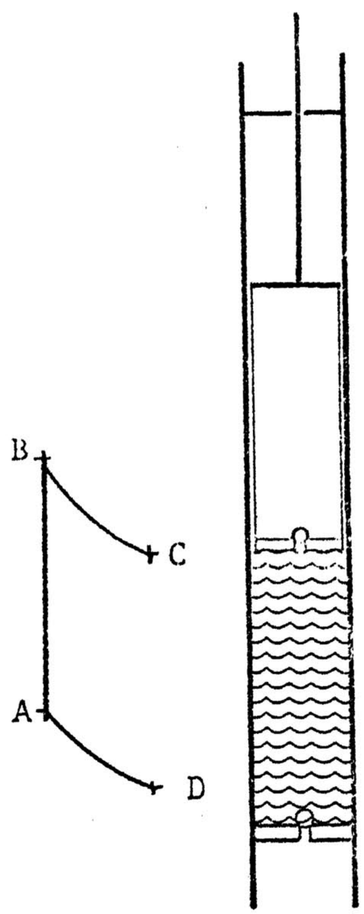


Figure 6.

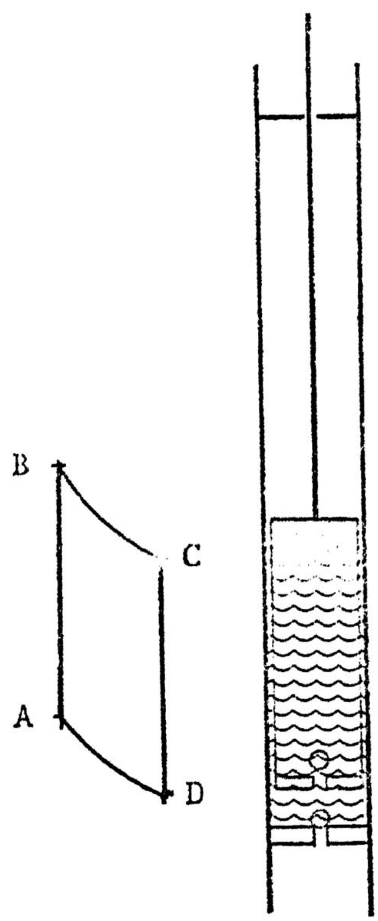


Figure 7.

It is difficult to write a suitable boundary condition to describe this pump motion. However, it is possible to write an equation, (15), that can describe all pump movement including plunger sticking.

$$\alpha u_{L,t} + \beta \frac{\partial u}{\partial x}\Big|_{L,t} = P(t) \tag{15}$$

Appropriate choices of α, β, and $P(t)$ will simulate pump performance between any two pump positions.

Between D and A, Figure 4, the rods are being loaded with the weight of the fluid column above the pump. The plunger load is a function of the pressure in the pump. The pump pressure will be a function of the volume of the pump chamber. At this condition:

where: $D \leq u_{L,t} \leq A$

$\alpha = o$

$\beta = 1$

and $P(t) = G_1(u_{L,D} - u_{L,t})$.

G_1 is a function describing the fluid loading of the plunger. The boundary-condition would reduce to:

$$\frac{\partial u}{\partial x}\Big|_{L,t} = G_1(u_{L,D} - u_{L,t}) \tag{16}$$

The rod strain is represented as a function of the plunger displacement.

Between points A and B the entire weight of the fluid column above the plunger is borne by the rods. Therefore, the rod strain at the pump is constant and equal to the weight of the fluid divided by Young's modulus of elasticity. Constants for the boundary condition are:

$\alpha = o$ where: $A \leq u_{L,t} \leq B$

$\beta = 1$

$P(t) = Wt_{fluid}\ (E \cdot A_{rod})$ and the boundary condition becomes:

$$\frac{\partial u}{\partial x}\bigg|_{L,t} = \frac{Wt_{fluid}}{E \cdot A_{rod}} \tag{17}$$

Between points B and C the rods are unloading the fluid weight and the unloading is a function of the pressure-volume characteristics of the gas in the pump chamber. Boundary condition variables are:

while $B \leq u_{L,t} \leq C$

$\alpha = o$

$\beta = 1$

$$P(t) = \frac{Wt_{fluid}}{E \cdot A_{rod}} - G_2(u_{L,t} - u_{L,B})$$

Where G_2 describes the expansion of the gas as a function of a plunger position, the boundary condition becomes:

$$\frac{\partial u}{\partial x}\bigg|_{L,t} = \frac{Wt_{fluid}}{E \cdot A_{rod}} - G_2(u_{L,t} - u_{L,B}) \tag{18}$$

Between points C and D the rods are unloaded and the tubing supports the fluid column. During this part of the stroke there is no strain since the traveling valve is freely passing fluid through the plunger. Therefore: while $C \leq u_{L,t} \leq D$

$\alpha = o$

$\beta = 1$

$P(t) = o$ and the boundary condition becomes

$$\frac{\partial u}{\partial x}\bigg|_{L,t} = o \tag{19}$$

If the pump was stationary at some point u_s then:

$$\alpha = 1$$

$$\beta = o$$

$$P(t) = u_s \tag{20}$$

and the boundary condition becomes: $u_{L,t} = u_s$

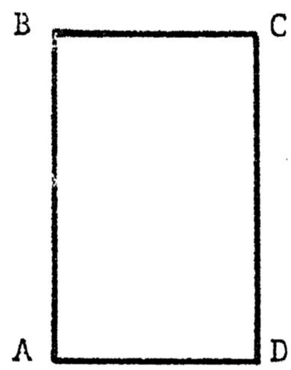


Figure 8. Ideal Pump Card

Figure 8 illustrates a dynamometer card for a one-hundred percent efficient pump. In the computer solution presented here, this is the pump action assumed. Notice that points A and D occur at the same plunger displacement. Points B and C also occur at identical plunger positions. This is accomplished by defining $u_A = u_D$ and $u_C = u_B$.

CHAPTER III

THE DERIVATION OF FINITE DIFFERENCE ANALOGIES TO THE ANALYTIC EQUATIONS

This paper explores two uses of the differential equation developed in Chapter II. First, it verifies Gibbs' experience in simulating pumping systems using finite difference analogies to the differential equation and its boundary conditions. Second, the finite difference analogy to the differential equation of motion is used to predict bottom-hole behavior from surface data. To accomplish both tasks, finite representations of the equation of motion and the boundary equation must be developed.

Representation of the Equation of Motion

The finite difference analogs are based on the Taylor's series expansion of a function. To represent the second derivative of displacement with respect to axial distance, the Taylor's series representation of displacement at the i + 1 and i - 1 nodes is written centered at "i".

$$u_{i+1,n} = u_{i,n} + \Delta x \left.\frac{\partial u}{\partial x}\right|_{i,n} + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x}\right|_{i,n} + \frac{\Delta x^3}{3!} \frac{\partial^3 u}{\partial x^3} + \cdots$$
$$\cdots + \frac{\Delta x^m}{m!} \frac{\partial^m u}{\partial x^m} + \cdots \qquad (21)$$

$$u_{i+1,n} = u_{i,n} - \Delta x \frac{\partial u}{\partial x}\bigg|_{i,n} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2}\bigg|_{i,n} - \frac{\Delta x^3}{3!} \frac{\partial^3 u}{\partial x^3} + \cdots$$

$$\cdots + \frac{(-1)^m \Delta x^m}{m!} \frac{\partial^m u}{\partial x^m} + \cdots \tag{22}$$

Where i is the subscript for axial distance and n is the time subscript If Equations (21) and (22) are added and all terms of fourth and higher orders are omitted, Equation (23) is obtained.

$$u_{i+1} + u_{i-1,n} = 2u_{i,n} + \Delta x^2 \frac{\partial^2 u}{\partial x^2}\bigg|_{i,n} \tag{23}$$

Equation (23) is rearranged so that an expression for the second derivative of the displacement with respect to axial distance is obtained.

$$\frac{\partial^2 u}{\partial x^2}_{i,n} = \frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n}}{\Delta x^2} \tag{24}$$

Equation (24) is centered at node i,n and is third order correct.

By similarly expanding two Taylor's series in the time dimension about the node i,n a third order correct expression for the second time derivative can be obtained.

$$\frac{\partial^2 u}{\partial t^2}_{i,n} = \frac{u_{i,n+1} - 2u_{i,n} + u_{i,n-1}}{\Delta t^2} \tag{25}$$

The first time derivative of displacement can be represented as:

$$\frac{\partial u}{\partial t}_{i,n} = \frac{u_{i,n+1} - u_{i,n}}{\Delta t} \tag{26}$$

This is first order correct equation but is chosen because higher order correct analogs lead to unstable solutions in the heat equation and such a choice conforms with the usual convention.

To complete the finite difference analogy of the equation of motion, Equations (24), (25) and (26) are substituted into Equation (6):

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} - \frac{\pi\sqrt{E/\rho}}{2L}\nu\frac{\partial u}{\partial t} \tag{6}$$

Substitution of (24), (25) and (26) gives:

$$\frac{u_{i,n+1} - 2u_{i,n} + u_{i,n-1}}{\Delta t^2} = \frac{E}{\rho}\left(\frac{u_{i+1,n} - 2u_{i,n} + u_{i-1}}{\Delta x^2}\right) - \frac{\pi\sqrt{E/\rho}}{2L}\nu\left(\frac{u_{i,n+1} - u_{i,n}}{\Delta t}\right) \tag{27}$$

Equation (6) can be written as:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} + \frac{\pi\sqrt{E/\rho}}{2L}\nu\frac{\partial u}{\partial t} \tag{28}$$

This equation can be classed as a linear hyperbolic equation. Mickley et al [6], Lapidus [7], and Ralston and Wilf [8], show that for the absolutely hyperbolic partial differential equation (29)

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{29}$$

an explicit difference form (30) will be

$$\frac{u_{i,n+1} - 2u_{i,n} + u_{i,n-1}}{\Delta t^2} - \left(\frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n}}{\Delta x^2}\right) \tag{30}$$

stable if the correct ratio of Δt to Δx, $\quad \frac{\Delta t}{\Delta x} \leq 1 \qquad (31)$

Equation (31) is maintained. Douglas[5] and the above references 6, 7, and 8, imply that the stability of such a scheme is unaffected by the choice of analogy for the first order derivative.

Hildebrand[3] and all references 5, 6, 7, and 8 state that existing evidence indicates that stability implies convergence. Therefore, the author assumes that a stable solution is a convergent solution.

Using Equation (31) as criteria for stability, Δt is chosen so that:

$$\Delta t = \frac{\Delta x}{\sqrt{E/\rho}} \tag{32}$$

and r is defined by:

$$r = \frac{\pi \sqrt{E/\rho}}{2L} \nu \ \Delta t \tag{33}$$

Equation (28) can be rearranged to give:

$$u_{i,n+1}(1+r) = u_{i+1,n} + u_{i-1,n} - u_{i,n-1} + r\, u_{i,n} \tag{34}$$

Division by (1+r) gives an explicit expression for $u_{i,n+1}$.

$$u_{i,n+1} = \frac{u_{i+1,n} + u_{i-1,n} - u_{i,n-1} + r\, u_{i,n}}{(1+r)} \tag{35}$$

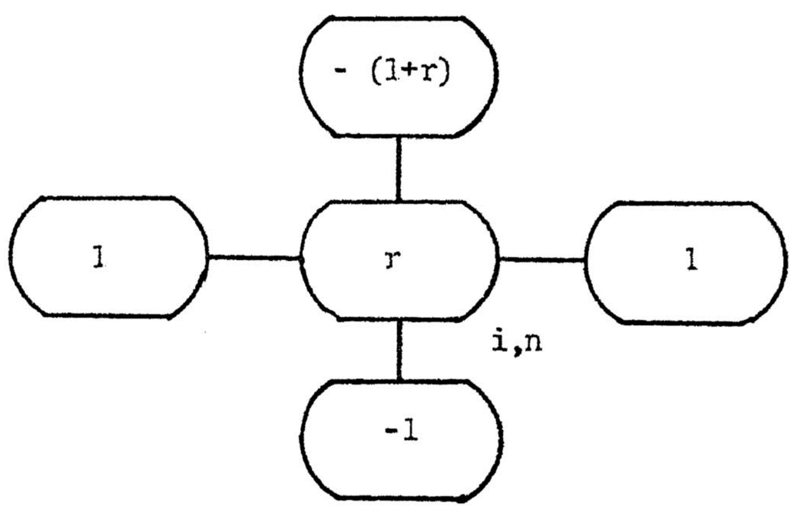


Figure 9. Computational Molecule for Difference Analog

Figure 9 illustrates the computation molecule for Equation (34).

Boundary Condition Analogs

Before pumping action can be simulated, the boundary conditions must be represented in a form suitable for digital computation. The surface boundary condition requires no approximation. The movement of the polished rod is described by Equation (12) and

$$u_{1,\theta} = L_5 \left[\sin^{-1}\left(\frac{L_1 \sin \theta}{h}\right) + \cos^{-1}\left(\frac{L_3 + h^2 - L_4^2}{2L_3 h}\right)\right] \qquad (12)$$

θ is a function of time described by Equation (14).

$$\theta = \frac{S.P.M.\ (2\pi)}{60}\ t \qquad (14)$$

where t is in seconds.

The initial conditions, Equations (7) and (8), define the initial veolcity and dynamic stress. The initial velocity is defined by (7).

$$\left.\frac{\partial u}{\partial t}\right|_{x,o} = o \qquad (7)$$

$\frac{\partial u}{\partial t}$ is expressed in finite difference form by Equation (26). To make Equation (26) zero:

$$u_{i,o} = u_{i,-1} = u_{i,-2} \qquad (36)$$

This allows suitable filling of the computation molecule. Equation (8) expresses the absence of initial dynamic stresses.

$$\left.\frac{\partial u}{\partial x}\right|_{x,o} = o \qquad (8)$$

If there is no change of displacement with axial distance, then all displacements must be equal.

$$u_{i,o} = u_{i,o} \qquad 1 \leq i \leq L \tag{37}$$

The pump boundary condition, Equation (15), is complicated by the presence of a derivative term

$$\alpha\, u\,(L,t) + \beta\, \frac{\partial u}{\partial x}_{(L,t)} = P(t) \tag{15}$$

that must be evaluated. The usual method of expanding two Taylor's series about the point of interest is followed but here the point of interest is the plunger end of the rod string.

$$u_{L-\Delta x,t} = u_{L,t} - \Delta x \frac{\partial u}{\partial x}_{L,t} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{2x^2}_{L,t} - \frac{\Delta x^3}{3!} \frac{\partial^3 u}{\partial x^3} \;--- \tag{38}$$

$$u_{L-2\Delta x,t} = u_{L,t} - 2\Delta x \frac{\partial u}{\partial x}_{L,t} + \frac{4\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2}_{L,t} - \frac{8\Delta x^3}{3!} \frac{\partial^3 u}{\partial x^3}_{L,t} \tag{39}$$

Both Taylor's series, Equations (38) and (39), are truncated after the second order terms. All remaining terms of (38) are multiplied by four and the remainder of (39) is subtracted from the result.

$$4u_{L-\Delta x,t} - u_{L-2\Delta x,t} = 3u_{L,t} - 2\Delta x \frac{\partial u}{\partial x}_{L,t} \tag{40}$$

Equation (40) can be written explicitly as a second order accurate representation of $\frac{\partial u}{\partial x}$.

$$\frac{\partial u}{\partial x}_{L,t} = \frac{3/2u_{L,t} - 2u_{L-\Delta x,t} + 1/2u_{L-2\Delta x,t}}{\Delta x} \tag{41}$$

The pump boundary condition now becomes:

$$\alpha\ \dot{i}_{L,t} + \beta\left(\frac{3/2u_{L,t} - 2u_{L-\Delta x,t} + 1/2u_{L-2\Delta x,t}}{\Delta x}\right) = P(t) \tag{42}$$

The plunger displacement can be explicitly expressed as:

$$u_{L,t} = \frac{\Delta x\ P(t) + u_{L-\Delta x,t}(2\beta) - 1/2u_{L} - 2\Delta x,t(\beta)}{(\alpha\Delta x + 3/2\ \beta)} \tag{43}$$

Equation (43) completes the expression of analytic equations in finite difference terms.

Using Equation (34) to predict bottom-hole behavior from surface data is somewhat different. The data that is known includes surface displacement as a function of time and the polished rod load (or surface stress) as a function of time. Therefore, two boundary conditions are known. However, time conditions do not exist. It is possible to rewrite Equation (34) as:

$$u_{i+1,n} = (1+r)\ u_{i,n+1} - r\ u_{i,n} + u_{i,n-1} - u_{i-1,n}\ . \tag{44}$$

Equation (44) can be used to predict values ahead in the space direction and if the time values are all known as the i location, then values at i+1 can be determined. It is appropriate to point out that the criteria for stability now becomes

$$\frac{1}{\alpha}\frac{\Delta x}{\Delta t} \leq 1 \tag{45}$$

and t can still be chosen by Equation (32)

$$\Delta t = \frac{\Delta x}{\sqrt{E/\rho}} \tag{32}$$

However, if:

$$\Delta t \leq \frac{\Delta x}{\sqrt{E/\rho}} \tag{46}$$

the resulting difference equation will be unstable.

After appropriate displacement and polished rod data for x = o are obtained for all times of interest, the polished rod load can be converted to strain easily and a first order correct boundary condition can be written:

$$\frac{u_{1,n} - u_{o,n}}{\Delta x} = \frac{(PRL + W_{rod\ in\ fluid})}{E \cdot A_{rod}} \qquad (47)$$

The displacement of $u_{o,n}$ and $u_{1,n}$ determined from the displacement and polished rod histories and Equation (47) can then be used to start the calculations with Equation (44) down the rod. The calculations are simply stopped at the lower end of the rod and no further boundary conditions are needed.

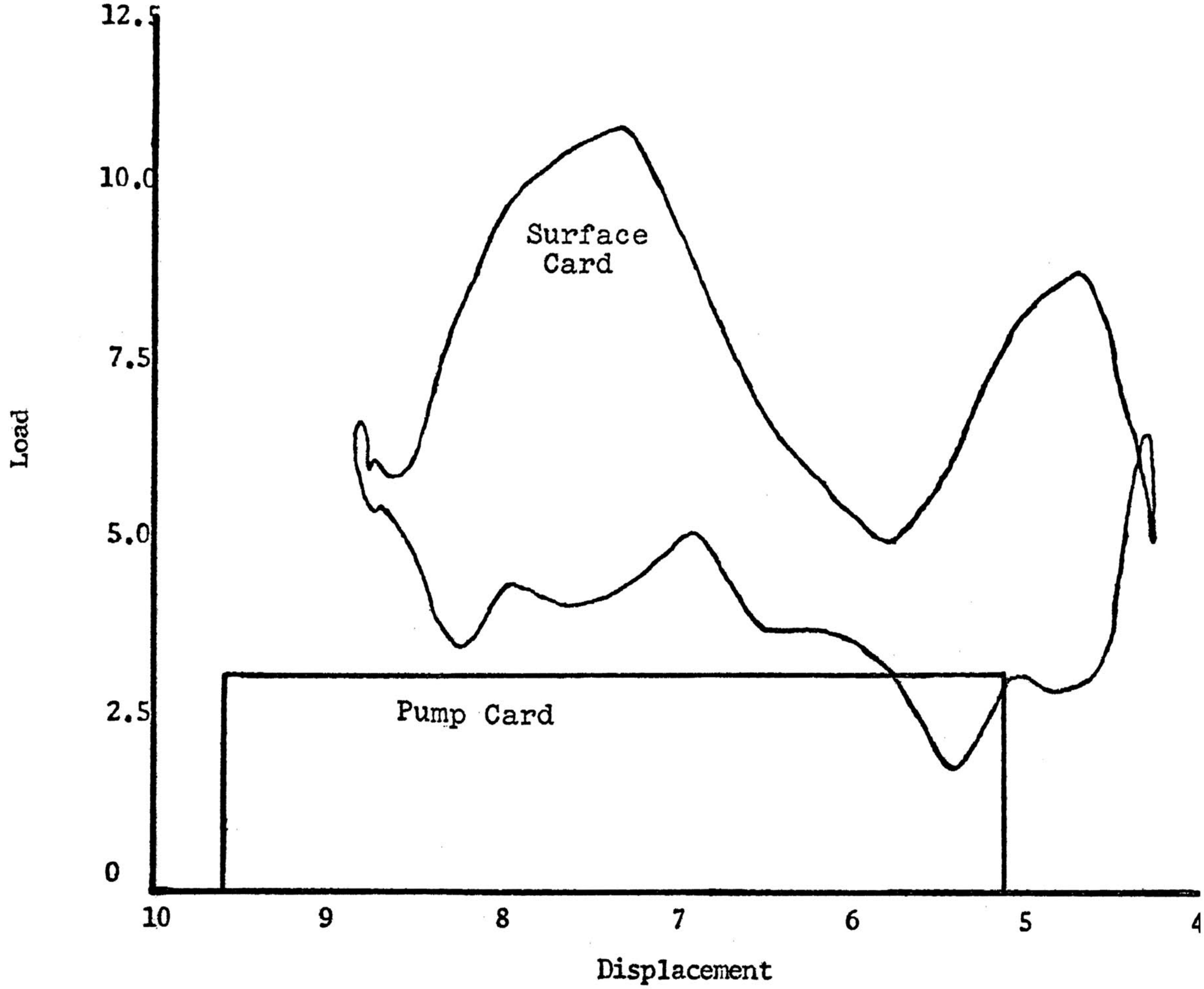
CHAPTER IV

RESULTS AND CONCLUSIONS

Figure 10 is a surface dynagraph card computed from the program simulating sucker-rod pumping systems. The physical description of the pumping system is given in the Figure. The dynagraph card has the usual shape and can be considered to accurately represent the behavior of a sucker-rod pumping system.

The simulation program could be used to design pumping systems. Using it for small, shallow wells would not justify the expense. However, for deep wells or wells with a high volume of liquid production, the use of such a system for design would be feasible. The simulation program could be utilized to check the present published sucker-rod pumping design criteria. If the results warranted, the present criteria could be improved using this program. A simulation program such as this is the only technique now available to aid in the design of dual zone pumping systems.

Figure 11 is a pump dynagraph card calculated from output of the simulator program. Polished rod loads and positions as functions of time were input to the analyzer program. This data was then used to calculate the behavior of the pump fastened to the bottom of the sucker-rod string. The pump cards in Figure 10 and Figure 11 are identical.



Total Rod Length	=	3000 feet
Specific Gravity of Pumped Fluid	=	0.90
API Pump Unit Size (Standard Design)	=	114
Stroke Length	=	54 inches
Strokes per Minute	=	18 SPM
Plunger Diameter	=	2 inches
Tubing Diameter	=	2½ inches
Rod Diameter	=	3/4 inches

Figure 10. Synthetic Polished Rod Card for 100% Efficient Pump

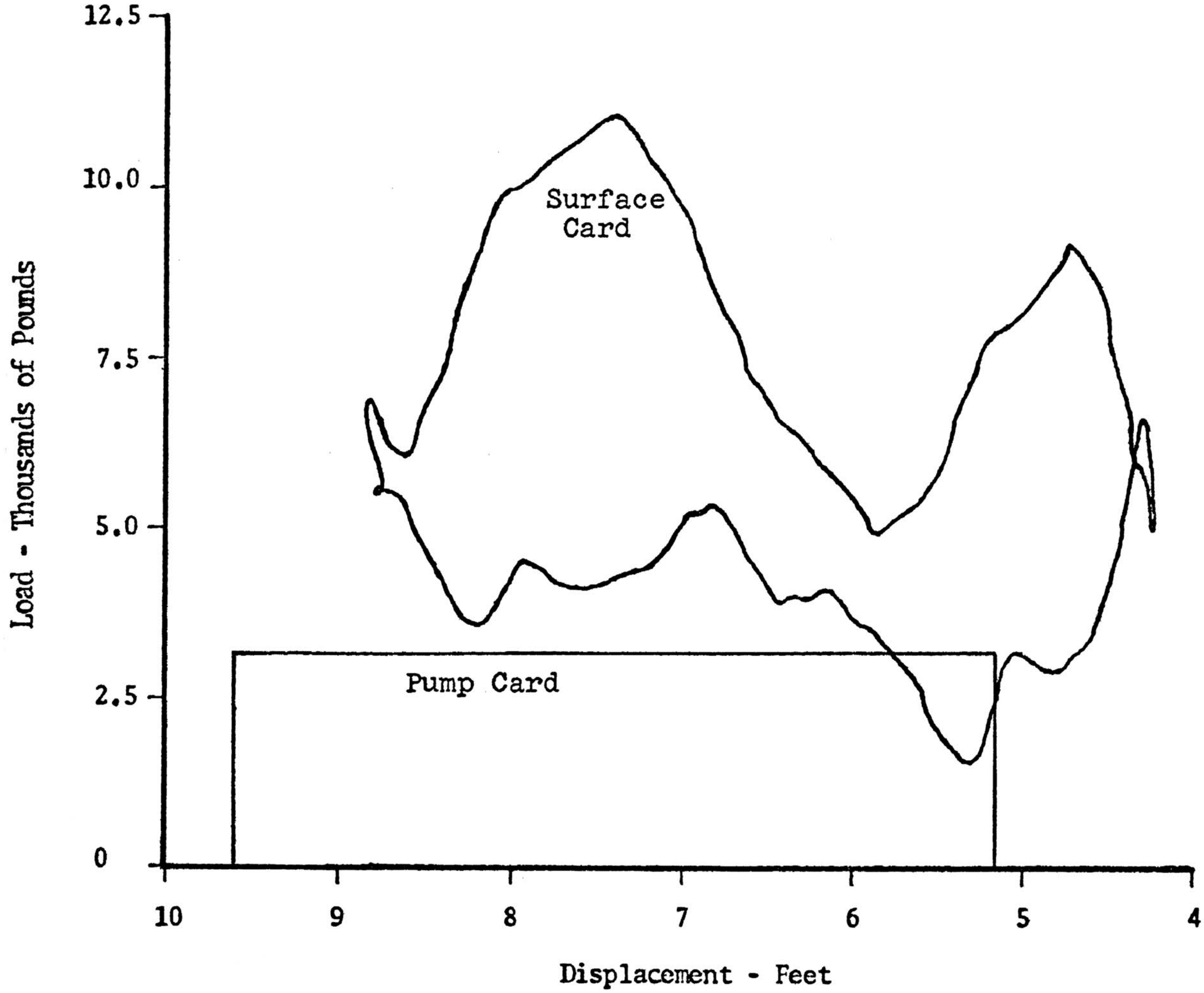


Figure 11. Pump Card Predicted from Surface Card

This is sufficient to prove that the variables of distance and time can be interchanged and the problem of determining bottom-hole pump behavior from surface data can be solved. The problem to be solved becomes an initial value problem if we consider that knowing the positions and derivatives with respect to axial distance for all values of time at the top of the sucker-rod is analogous to knowing two initial conditions. This then permits proceeding down the rod calculating displacements for all times at the new value of axial distance. This procedure is continued until the end of the rod string is reached. Rod load is determined by evaluating the strain and resulting load at any point of interest.

The intent of this paper was not to produce a sucker-rod pump evaluator but merely to prove that it was possible to generate down-hole pump behavior from polished rod data. A program to analyze sucker-rod pumping systems makes it possible to compute a bottom-hole pump card and analyze its efficiency rather than attempt to analyze the surface cards with their highly irregular shapes. This would greatly simplify the problem of analyzing pumping systems.

There is still a great deal of work undone. Before a general sucker-rod evaluator can be produced, the dimensionless damping factors must be determined. It is not obvious if it can be predicted from the pump design and knowledge of the pumped fluid or if it is an empirical factor that can only be determined from trial and error. Gibbs states that "the dimensionless damping factors vary over small ranges for all known pumping systems". However, he does not provide any information to the magnitude of these ranges or how his statement can be proven.

The techniques described in this paper can be readily extended to tapered rod strings. Instead of eliminating the radius of the rod in the differential equation, it would be necessary to include this as a function of axial distance. Since the difference equations are well behaved, such an addition would not be expected to cause undue difficulty in obtaining solutions of the new equations. The extension of the simulator program to dual zone pumps would simply require adding another set of pump equations at the high zone. The analyzer program would require no modification to account for multiple zone pumps.

The use of the analyzer program presupposes knowing both polished rod displacement and load as a function of time. Most surface dynagraph data is made on a displacement versus load basis. Dynagraphs are available, however, which can record surface data directly versus time. If this is not possible, some scheme for translating displacement versus load to a displacement and load versus time basis must be devised. I think this can be done by making a four bar linkage analysis of the surface pumping unit and knowing the torque characteristics of the prime mover. The use of the analysis program would provide the producer with a low cost means of analyzing his pumping systems. No longer would it be necessary to intuitively gaze at the surface dynagraph card and using some optical-mental process, hypothesize what the bottom-hole card would look like. Instead, a surface card could be analyzed and input to the analyzer program to produce a bottom-hole pump card. This card could then be analyzed directly.

APPENDIX A

STATIC EFFECTS AND LOAD CALCULATIONS

Chapter III describes the mathematical tools used to approximate the motion of the sucker-rod pumping system. However, those equations describe only the dynamic motion of the system. Before they can be used, the static conditions mentioned in Chapter II must be superimposed on the dynamic solution. To relate the analysis to available dynagraph data, a method must be devised for calculating the polished rod forces present.

The dynamic loads at any point in the sucker-rod string can be calculated using Hooke's Law as shown in Chapter III. Superimposed on these forces is the rod stretch due to gravity. The total loads are the sum of the dynamic loads and the static load. Thus, the polished rod load is the sum of the dynamic load at the top of the sucker-rod and the bouyant weight of the rods.

$$PRL = F_{o,t} + W_b \tag{48}$$

The pump load is merely the dynamic load at the bottom of the rod.

$$P_L = F_{L,t} \tag{49}$$

The displacement at the pump is also affected by gravity. The actual pump displacement is the sum of the dynamic displacement and the static rod stretch.

$$PD = u_{L,t} + \text{Static Stretch} \tag{50}$$

At any point of interest the same calculations could be performed to calculate the standard form of dynagraph card. Typical points of interest would be the junction points in a tapered rod string.

APPENDIX B

PROGRAM DESCRIPTIONS

Dynamic Sucker-Rod Analysis Simulator Program

The program computes displacements by retaining the displacements from the two most recent time steps. The displacements computed fill the array UNEW(I), the displacements held in historical order are UOLD(I) and U2LD(I). The program is initiated by assuming that the pump is approaching its lowest position with a full load of fluid. The initial displacement of the polished rod is computed and from Equations (32) and (33) in Chapter III all other elements of the UNEW(I), UOLD(I) and U2LD(I) arrays are set equal to UNEW(1). Since it requires N-1 time increments for a force to travel the length of the rod, all elements of UNEW(I), UOLD(I) and U2LD(I) less than I(Δx) are held equal to the initial UNEW(I). This process is continued until the number of time steps are equal to the number of distance increments. At this point the first force impulse will have had the opportunity to travel to the end of the rod string. At this time all new displacements are calculated using Equation (31) and the plunger displacement using Equation (39). Once the bottom displacement is calculated, the down-hole conditions are checked to determine the status of the pump boundary condition. Then the constants α, β and P(t) are fixed.

Once setting the pump boundary condition is complete, the values of the dynamic and true forces at the plunger and polished rod are calculated. The program writes output each five time steps. The values of time, crank angle, dynamic displacements at the top, middle and bottom of the rod string are contained in this output. The true pump displacement, the dynamic forces and true loads at the top and bottom of the rod string are also printed. If output of the displacements of all axial nodes is desired, a temporary variable IOP is read into the program with a value of o.

After the output is completed or before the five cycles are complete, the counting variable is incremented by one. All the values in the UZLD array are replaced with the values of the UOLD array. The UOLD values assume the UNEW values. If the program has run fewer cycles than the desired number, control is transferred to the computation of UNEW(I) and the cycle is begun all over again.

Sucker-Rod Analysis Program

The current program consists of a skeleton of calculation. Physical characteristics of the rod string are read and the linear constants of the difference equations are calculated. The values of displacement and polished rod as a function of time are read (this is now accomplished at each Δt). It is conceivable that these could be replaced by a continuous function if this could be fit. However, the discrete form is very convenient for computational purposes. The program then uses Equation (44) to calculate the displacement at succeeding values of distance for each value of time. Static displacements and forces are added to these calculated dynamic effects where appropriate. The values of the input dynagraph card and the resulting pump card are then written and plotted to construct the pump dynagraph card.

APPENDIX C

FORTRAN LISTING OF SIMULATOR PROGRAM

```
C     DYNAMIC ANALYSIS OF SUCKER ROD PUMPING SYSTME  RMK
      ARCSNF(ZZ)=  ATAN(ZZ/( SQRT(1. - ZZ*ZZ))  )
      ARCCSF(ZZ) =  ATAN( SQRT(1.-ZZ*ZZ)/ZZ)
      DIMENSION UOLD(101),U2LD(101),UNEW(101)
C
      READ 10,XL1,XL2,XL3,XL4,XL5
   10 FORMAT(5F6.2)
      READ 11,TL,RHO,XNU,E,AR,GR,D,AP
   11 FORMAT(2F8.1,F7.3,F10.0,F6.3,F6.3,F8.1,F7.4)
    6 FORMAT(4F6.3)
      READ 6,R1,R2,R3,R4
      READ 6,WM1,WM2,WM3,WM4
      READ 6,A1,A2,A3,A4
  121 READ 101,SPM,T,N,BTHTA,IOP,NEND
  101 FORMAT(2F7.3,I4,F7.1,I2,I2)
C
      BTHTA = BTHTA *6.2831853 + 6.2831853 *SPM * T/60.
      WROD = TL*(WM1*R1 + WM2*R2 + WM3*R3 + WM4*R4)
      FB = WROD*GR*62.4/490.
      WB = WROD - FB
      RSTH = WB*TL/(2.*E*AR)
      WF = .433* GR*TL*AP - FB
      A =  SQRT(E* 144.* 32.2 / RHO)
      JUDI = 1
      XN = N - 1
      DELX = TL/XN
      DELT = DELX/A
      P = 3.1415927*A*XNU*DELT/(2.*TL)
C     INITIALIZATION
      ALPH = 0.
      BETA = 1.0
      PT = 0.0
      G1 = WF/(E*AR)
      G2 = G1
C     WE WILL NEED SOME SFUNCTIONS TO DEFINE G1 AND G2 HERE
      MID = (N -1)/2
      NM1 = N - 1
      NPS = 0
      KOUNT = 0
      PRINT 2,RSTH,WB,WF,WROD,DELX,DELT
    2 FORMAT(6H RSTH=,F7.4,5H  WB=,E11.4,5H  WF=,E11.4,7H  WROD=,E11.4,
     17H  DELX=,E11.4,7H  DELT=,E11.4 )
      PRINT 4
    4 FORMAT(31H-  TIME    THTA     DISPLACEMENTS,20H IN FEET  AT THE     ,
     132HDYNAMIC FORCES IN LBS.F  LOAD IN,14HLBS.F  AT THE , /7H   SEC ,
     240H  RADIANS  POL.ROD MIDDLE  BOTTOM    PUMP,18H          TOP     ,
     332H BOTTOM       POL.ROD          PUMP/)
C     CALCULATION OF SURFACE BOUNDARY CONDITION BEGINS HERE
C
  701 THTA = 6.2831853/60. * SPM * T
      H =  SQRT(XL1*XL1 + XL2*XL2 + 2.*XL1*XL2* COS(THTA))
      ARGU1 = XL1* SIN(THTA)/H
      ARGU2 = (H*H + XL3*XL3 - XL4*XL4)/(2.*XL3*H)
      ZETA = ARCCSF(ARGU2)
```

SOURCE STATEMENT

```
      XI = ARCSNF(ARGU1)
C
  906 UNEW(1) = XL5*( XI + ZETA)
      IF(KOUNT) 17,16,17
   16 CONTINUE
      SAM = UNEW(1)
      DO 20  I = 1,N
      UOLD(I) = UNEW(1)
   20 U2LD(I) = UNEW(1)
   17 CONTINUE
C
      IF(JUDI - NM1) 750,755,760
  750 JUDI = JUDI + 1
      JOAN = JUDI + 1
      DO 752 I = JOAN,N
  752 UNEW(I) = SAM
      DO 30  I = 2,JUDI
   30 UNEW(I)=(UOLD(I+1)+ UOLD(I-1)- U2LD(I)+R*UOLD(I))/(1.+R)
      GO TO 765
C
  755 DO 300 I =2,NM1
  300 UNEW(I)=(UOLD(I+1)+ UOLD(I-1)- U2LD(I)+R*UOLD(I))/(1.+R)
      UNEW(N) = SAM
      JUDI = JUDI + 1
      GO TO 765
  760 DO 350  I = 2,NM1
  350 UNEW(I)=(UOLD(I+1)+ UOLD(I-1)- U2LD(I)+R*UOLD(I))/(1.+R)
C
      UNEW(N)=(DELX*PT+2.*BETA*UNEW(N-1)-.5*BETA*UNEW(N-2))/(ALPH*DELX
     1+ 1.5*BETA)
C
C     CHECKING ROUTINE STARTS HERE
C
      DUDX = 1.5*UNEW(N) - 2.*UNEW(N-1) + .5*UNEW(N-2)
      IF( ABS(DUDX) - 0.00001 ) 50,50,60
   60 IF (DUDX) 50,50,61
   50 IF(UNEW(N) - UOLD(N)) 58,55,55
   55 ALPH =0.0
      BETA = 1.0
      PT =0.0
      GO TO 99
C
   58 ALPH = 1.0
      BETA = 0.0
      PT = UNEW(N)
      GO TO 99
C
   61  IF(E*AR*DUDX/DELX - WF) 70,64,64
   70  IF( ABS(E*AR*DUDX/DELX - WF) - 5.) 64,64,99
   64 ALPH=0.
      BETA =1.
      PT = WF/(E*AR)
      IF(UNEW(N) - UOLD(N)) 99,99,66
   66 ALPH = 1.0
      BETA = 0.0
```

```
      SOURCE STATEMENT
      PT = UNEW(N)
      GO TO  99
C
  765 CONTINUE
C
   99 FDT =B*AR/DELX*(-1.0*UNEW(1)+    UNEW(2)              )
      PRL = FDT * NB
      FDB = B*AR/DELX*(1.5*UNEW(N) - 2.*UNEW(N-1) + .5*UNEW(N-2))
      TRULD = FDB - WB
      PDM = UNEW(N) + RSTH
      IF(THTA+9.424878-BTHTA) 96,399,399
  399 PUNCH 8,T,UNEW(1),PRL
    8 FORMAT(3E16.8)
      NPS = NPS + 1
      IF(KOUNT - 5)96,94,94
   94 PRINT 7,T,THTA,UNEW(1),UNEW(MID),UNEW(N),PDM,FDT,FDB,PRL,TRULD
    7 FORMAT(1X,F7.3,1X,F7.3,2X,F7.3,1X,F7.3,1X,F7.3,1X,F7.3,2X,E11.4
     1,E11.4,2X,E11.4,1X,E11.4 )
      KOUNT = 0
      IF(IOP) 96,28,96
   28 DO 26  I = 1,N
   26 PUNCH 27,I,UNEW(I)
   27 FORMAT(15X,7H UNEW( ,I3,6H ) =  ,F9.4)
      PUNCH 25,H,XI,ZETA,DUDX
   25 FORMAT(4H-H= ,E11.4,5X,6H XI=   ,E11.4,5X,7HZETA= ,E11.4,7HDUDX
     11E11.4 )
   96 CONTINUE
   93 KOUNT = KOUNT + 1
   98 T = T + DELT
      DO 97 I=1,N
      U2LD(I) = UOLD(I)
   97 UOLD(I) = UNEW(I)
      IF(THTA - BTHTA)701,701,100
  100 CONTINUE
      PRINT 9,NPS
    9 FORMAT(I6)
      IF(NEND - 1) 121,104,104
  104 CONTINUE
      END
```

APPENDIX D

FORTRAN LISTING OF ANALYZER PROGRAM

SOURCE STATEMENT

```
C         EXPLICIT SOLUTION OF WAVE EQUATION--CENTERED BACKWARD IN DISTANCE
C         USE TIME LEVELS N+1,N,N-1 AND DISPLACEMENTS AT I,I-1 TO FIND
C         DISPLACEMENT AT I+1,N.
C          DYNAMIC SUCKER ROD PREDICTOR= DSRPRED
          DIMENSION U(25,500),PRL(500),T(500)
          READ(5,11) TL,RHO,XNU,E,AR,GR,D,AP
       11 FORMAT(2F6.1,F7.3,F10.0,F6.3,F6.3,      F8.1,F7.4)
          READ(5,22) N,WM1,R1,NPS
       22 FORMAT (I4,2F6.3,I4)
          WROD = TL*WM1*R1
          FB = WROD*GR*62.4/490.
          WB = WROD   - FB
          RSTH = WB*TL/(2.*E*AR)
          WF = .433*GR*TL*AP-FB
          A = SQRT(E*144.*32.2 /RHO)
          DELX = TL/FLOAT(N-1)
          DELT = DELX/A
          R = 3.1415927*A*XNU*DELT/(2.*TL)
          DO 100   J = 1,NPS
          READ(5,555) T(J),U(1,J),PRL(J)
      555 FORMAT(3E16.8)
      100 U(2,J) = (PRL(J) - WB)*DELX/(E*AR) + U(1,J)
          JSTART = 2
          NM1 = N- 1
          DO 250   I = 2,NM1
          NPS = NPS - 1
          DO 200   J = JSTART,NPS
      200 U(I+1,J) = (1.+ R) * U(I,J+1) - R* U(I,J) + U(I,J-1) - U(I-1,J)
          JSTART = JSTART + 1
      250 CONTINUE
          DO 300   J=JSTART,NPS
          PRL(J)=E*AR/DELX*(U(2,J)-U(1,J)) + WB
          PMPLD = E*AR/DELX*(1.5*U(N,J)-2.*U(N-1,J)+ 0.5*U(N-2,J))-WB
          PDM = U(N,J) + RSTH
      300 WRITE(6,556) T(J),U(1,J),PRL(J),PDM,PMPLD
      556 FORMAT(F7.3,1X,F7.3,2X,E11.4,5X,F7.3,2X,E11.4)
          END
```

TABLE E

TABLE OF NOMENCLATURE

Algebraic Symbol	FORTRAN Symbol	Description	Units
α	A	Velocity of force propogation	feet/sec.
A_p	AP	Plunger Area	in.^{2}
A_{rod}	AR	Cross-sectional area of rod	in.^{2}
D	D	Differential depth of the pumped fluid	feet
E	E	Young's modulus of elasticity for the rod material	lbf/in.^{2}
G_r	GR	Specific gravity of fluid	
L	TL	Length of sucker-rod string	feet
L_1	XL1	Distance from center of crank to center of pitman bearing	feet
L_2	XL2	Distance from crankshaft center to saddle bearing center	feet
L_3	XL3	Distance from saddle bearing to tail bearing	feet
L_4	XL4	Length of pitman	feet
L_5	XL5	Distance from saddle bearing to center of polished rod	feet
M_1	WM1	Weight per foot of length of rods of cross-section A1	lb./feet
M_2	WM2	A2	
M_3	WM3	A3	
M_4	WM4	A4	
PD	PDM	Pump displacement	feet
PRL	PRL	Polished rod load	lb.
r	R	$\frac{\pi \alpha \nu}{2L} \Delta t$	

Algebraic Symbol	FORTRAN Symbol	Description	Units
R_1	R1	Fraction of TL made up of rods of cross-section A1	
R_2	R2	A2	
R_3	R3	A3	
R_4	R4	A4	
SPM	SPM	Plunger strokes per minute	
t	T	Time	seconds
u		Displacement of point (x) along rod	feet
$u_{i,n+1}$	UNEW(I)	Displacement at depth IΔX & it is now being calculated	feet
$u_{i,n}$	UOLD(I)	Old displacement at I$\cdot\Delta$X	feet
$u_{i,n-1}$	U2LD(I)	Displacement at one time step previous to n or UOLD(I)	feet
x		Axial distance along rod	feet
α	ALPH	Coefficient of $u_{L,t}$ in pump boundary condition	
β	BETA	Coefficient of $\frac{\partial u}{\partial x}_{L,t}$ in pump boundary condition	
Δt	DELT	Time increment	seconds
Δx	DELX	Interval distance along the sucker-rod axis	feet
ε		Strain	
θ	THTA	Crank angle of pump	radians
ν	XNU	Viscous damping factor	dimensionless
ρ	RHO	Density of rod material	lb./ft.^{3}
σ		Stress	lb_F/in.^{2}

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